# CS107: Type Checking/Inference and Functions

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# Recap

What did we learn last time?

- Compiling mutable variables and loops
- Syntax sugar
- Introduced type systems

# Our Grammar, Typed

```
<op>    ::= ['*' | '/' | '+' | '-' | '<' | '>' | '=' | '!']+
<type>  ::= <ident>                             // new
<bool>  ::= 'true' | 'false'
<atom>  ::= <number> | <bool> | '()'            // new
          | <ident>
          | '('<simp>')'
          | '{'<exp>'}'
<uatom> ::= [<op>]<atom>
<simp>  ::= <uatom>[<op><uatom>]*
          | 'if' '('<simp>')' <simp> ['else' <simp>]
          | <ident> '=' <simp>
<exp>   ::= <simp>[';'<exp>]
          | 'val' <ident> [':' <type>] '=' <simp>';'<exp> // optional type
          | 'var' <ident> [':' <type>] '=' <simp>';'<exp> // optional type
          | 'while' '(' <simp> ')' <simp>';'<exp>
```

# Inference Rules

**Typing judgments**: we write

$$\Gamma \vdash e : T$$

to assert that in the environment $\Gamma$, the expression $e$ is of type $T$.

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- $\Gamma$ is the **typing environment**: It stores knowledge about identifiers available at compile time, as a finite mapping from identifiers to types. Grammar:

  `Γ ::= ∅ | Γ, id:T`

- We write $\emptyset$ for the empty typing environment, and
- $\Gamma, id : T$ to extend the typing environment $\Gamma$ with a new mapping from $id$ to type $T$.

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**Inference rules** specify how we can form typing judgments:

$$\frac{\mathit{condition1} \qquad \mathit{condition2} \qquad \ldots}{\mathit{conclusion}}\ \textsc{Name of the rule}$$

If all conditions can be proven **true** then the conclusion is **true**.

# Inference Rules

1) Lit: $i$ is an Int, $b$ is a Boolean

$$\Gamma \vdash \texttt{Lit}(i) : \texttt{Int} \ \textsc{int} \qquad\qquad \Gamma \vdash \texttt{Lit}(b) : \texttt{Boolean} \ \textsc{boolean}$$

$$\Gamma \vdash \texttt{Lit}(\texttt{()}) : \texttt{Unit} \ \textsc{Unit}$$

We call inference rules without conditions **axioms**.

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We call inference rules without conditions **axioms**.

2) Unary: $op \in \{\texttt{"+"}, \texttt{"-"}\}$

$$\frac{\Gamma \vdash e : \texttt{Int}}{\Gamma \vdash \texttt{Unary}(op, e) : \texttt{Int}} \ \textsc{intunop}$$

# Inference Rules

3) Prim:

- $op \in \{\texttt{"+"},\ \texttt{"-"},\ \texttt{"*"},\ \texttt{"/"}\}$
- $bop \in \{\texttt{"=="},\ \texttt{"}\neq\texttt{"},\ \texttt{"}\leqslant\texttt{"},\ \texttt{"}\geqslant\texttt{"},\ \texttt{"<"},\ \texttt{">"}\}$

$$\frac{\Gamma \vdash e_1 : \texttt{Int} \qquad \Gamma \vdash e_2 : \texttt{Int}}{\Gamma \vdash \texttt{Prim}(op, e_1, e_2) : \texttt{Int}}\ \textsc{IntOp} \qquad \frac{\Gamma \vdash e_1 : \texttt{Int} \qquad \Gamma \vdash e_2 : \texttt{Int}}{\Gamma \vdash \texttt{Prim}(bop, e_1, e_2) : \texttt{Boolean}}\ \textsc{BoolOp}$$

# Inference Rules

4) Immutable variables

$$\frac{\Gamma \vdash e_1 : T_1 \qquad \Gamma, x : T_1 \vdash e_2 : T_2}{\Gamma \vdash \texttt{Let}(x, T_1, e_1, e_2) : T_2}\ \textsc{Let} \qquad\qquad \frac{\Gamma(x) = T}{\Gamma \vdash \texttt{Ref}(x) : T}\ \textsc{Ref}$$

# Examples

Prove that the following program is of type Boolean

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There is no more statement to prove! That means our initial statement was true.

# Type Checking and Type Inference

Prove that the following program is of type Boolean

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val x = 3; x == 4
```

Recall the grammar:

```
<exp>   ::= 'val' <ident> [':' <type>] '=' <simp>';'<exp>
          | ...
```

# Type Checking and Type Inference

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We write **???** for the placeholder/unknown type.

$$\texttt{Let}(\texttt{x}, \textbf{???}, \texttt{Lit}(3), \texttt{Prim}(\texttt{"}{=}{=}\texttt{"}, \texttt{Ref}(\texttt{x}), \texttt{Lit}(4)))$$

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Can we still do it? Yes, as only one rule can be applied to Lit(3).

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Can we still do it? Yes, as only the rule INT can be applied to Lit(3). This determines the instantiation of **???** with type `Int`.

# Type Checking and Type Inference

The type checking/inference step will be part of the semantic analyzer.

Key idea: types represent abstract values, and inference rules are the set of operations on these values.

The implementation of type checking/inference will structurally similar to the interpreter `eval`.

# Type Checking and Type Inference

We add a `Type` field in our AST now:

```scala
abstract class Type
case class BaseType(tp: String) extends Type
val IntType = BaseType("Int")
val BoolType = BaseType("Boolean")
val UnitType = BaseType("Unit")
object UnknownType extends Type

abstract class Exp {
  // ... Position
  var tp: Type = UnknownType
  def withType(pt: Type) = { tp = pt; this }
}
```

The type checker will have to resolve the type of each node.

## Type Checking and Type Inference

We are going to define two main functions:

The first thing is going to try to infer the type of `exp` in environment `env`. Type `pt` is a "suggestion" on what the type should be, but can be ignored. It returns an AST equivalent to `exp` with all types resolved.

```
def typeInfer(exp: Exp, pt: Type)(env: Env): Exp
```

# Inference Example

Example:

```
typeInfer(
  Let(x, UnknownType, Lit(3), Prim("==", Ref(x), Lit(4))),
  UnknownType // We don't have information at first
)(emptyEnv)
```

will return

```
Let(x, IntType,
  Lit(3),   /* tp == IntType */
  Prim("==",
    Ref(x), /* tp == IntType */
    Lit(4)  /* tp == IntType */
  )         /* tp == BoolType */
)           /* tp == BoolType */
```

The second is going to infer the type of `exp` and **verify that it conforms to type `pt`**. It also returns an equivalent AST with all types resolved.

```
def typeCheck(exp: Exp, pt: Type)(env: Env): Exp
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We need to define what "T1 conforms to T2" means.

T1 conforms to T2 if:

- T1 == T2, or
- T2 is UnknownType

# Implementation

```scala
// Check if 'tp' is well-formed. For now that means that 'tp'
// is not unknown
def typeWellFormed(tp: Type)(env: Env): Type = ...

// Check if 'tp' conforms to 'pt' and return the more precise type
// The returned type should also be well-formed
def typeConforms(tp: Type, pt: Type)(env: Env): Type = ...

def typeCheck(exp: Exp, pt: Type)(env: Env): Exp = {
  // First infer
  val nexp = typeInfer(exp, pt)(env)
  val rtp = typeConforms(nexp.tp, pt)(env)
  nexp.withType(rtp)
}
```

# Implementation

```
def typeInfer(exp: Exp, pt: Type)(env: Env): Exp = exp match
  case Lit(i: Int) => ???
  case Let(x, tp, rhs, body) => ???
  case ... => ...
```

# Implementation

```scala
def typeInfer(exp: Exp, pt: Type)(env: Env): Exp = exp match
  case Lit(i: Int) => ???            // Rule [Int]
  case Let(x, tp, rhs, body) => ??? // Rule [Let]
  case ... => ...
```

# Implementation

```scala
def typeInfer(exp: Exp, pt: Type)(env: Env): Exp = exp match {
  case Lit(i: Int) => exp.withType(IntType) // No conditions
  case Let(x, tp, rhs, body) => // Rule [Let]
    if (env.isDefined(x)) warn("reuse of variable name", exp.pos)

    // Left condition: env |- rhs: tp
    val nrhs = typeCheck(rhs, tp)(env)

    // Right condition: env, x:nrhs.tp |- body: pt (pt may be UnknownType)
    val nbody = typeCheck(body, pt)(env.withVal(x, nrhs.tp))

    // Conclusion
    Let(x, nrhs.tp, nrhs, nbody).withType(nbody.tp)
  case ... => ...
}
```

# Inference Rules (cont'd)

5) if:

$$\frac{\Gamma \vdash c_1 : \texttt{Boolean} \qquad \Gamma \vdash e_1 : T \qquad \Gamma \vdash e_2 : T}{\Gamma \vdash \texttt{If}(c_1, e_1, e_2) : T} \text{ IF}$$

6) Mutable variables

$$\frac{\Gamma \vdash e_1 : T_1 \qquad \Gamma, x : T_1 \vdash e_2 : T_2}{\Gamma \vdash \texttt{VarDec}(x, T_1, e_1, e_2) : T_2} \text{ VARDEC}$$

$$\frac{\Gamma(x) = T_1 \qquad \Gamma \vdash e_1 : T_1}{\Gamma \vdash \texttt{VarAssign}(x, e_1) : T_1} \text{ VARASSIGN}$$

# Inference Rules (cont'd)

7) while

$$\frac{\Gamma \vdash c_1 : \texttt{Boolean} \qquad \Gamma \vdash e_1 : \texttt{Unit} \qquad \Gamma \vdash e_2 : T_2}{\Gamma \vdash \texttt{While}(c_1, e_1, e_2) : T_2}\ \textsc{while}$$

# Interpretation With Types

```scala
abstract class Val
case class Cst(x: Any) extends Val

def eval(exp)(env: Env): Val = exp match {
  case Lit(i: Int) => Cst(i)
  case Prim(op, l, r) => evalPrim(op)(eval(l)(env), eval(r)(env))
  // ...
}
def evalPrim(op: String)(l: Val, r: Val) = (op, l, r) match {
  case ("+" , Cst(x: Int), Cst(y: Int)) => Cst(x + y)
  case ("==", Cst(x: Int), Cst(y: Int)) => Cst(x == y)
  // ...
}
```

# Compilation With Types

Assembly code does not have types. We need to make an "implementation" decision on how to represent the new types.

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- Boolean: 0 or 1, but we still use the full register.
- Unit: There are no operations using Unit, but for the ease of implementation, we still use the full register, and just let it hold uninitialized bits.

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Implementation of the operators:

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but X86 offers us a shortcut:

```
    set<op> %al          # set %al if flags validate <op>
                         # like jump, there are: sete, setne, setl, etc.
    movbq %al, %rax      # transform the byte into the full register
```

# Compilation With Types

We also have to modify our compilation for the `If` statements.

```
def trans(exp: Exp, sp: Int)(env: Env) = exp match {
  case If(cond, tBranch, eBranch) =>
    trans(cond, sp)(env)  // now reg at sp will contain 0 or 1
    transJumpIfTrue(sp)("if")
    // ...
```

What code would `transJumpIfTrue` generate?

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What code would `transJumpIfTrue` generate?

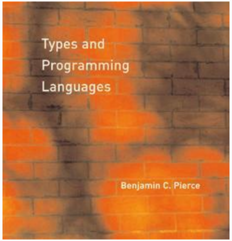
```
cmp ${regs(sp)}, $1    # INVALID syntax, only registers allowed.
je $label

test ${regs(sp)}, ${regs(sp)}
jnz $label
```

`test S, T` sets the flags accordingly to `S & T`: So if `sp` contains 1: `1 & 1` $\neq$ `0` so we jump (jnz). If `sp` contains 0: `0 & 0` $=$ `0`, then we don't jump.

# Further Reading

- Types and Programming Languages, Benjamin C. Pierce, 2002, MIT Press
- Bidirectional Typechecking, Jana Dunfield, Neel Krishnaswami, ACM Computing Surveys, 2019
- Propositions as Types, Philip Walder, Communication of ACM, 2015. Youtube vidoe: https://www.youtube.com/watch?v=IOiZatlZtGU



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